UNIT - 2

TWO DIMENSIONAL RANDOM VARIABLES

Discrete Random variable :-

Two Random variables x, y are said to be discrete if possible values are (x, y) are finite.

Let, x, y be two dimensional discrete Random variables then P(x, y) is called joint PMF, if

$$\boxed{\begin{array}{l} P(x,y) \geq 0 \\ \sum\sum P(x,y) = 1 \end{array}}$$

Marginal Distribution :-

Let x, y be two dimensional discrete random variable. Then the marginal distribution of x is

$$\boxed{P(x) = \sum_{y} P(x,y)}$$

The Marginal Distribution of y is

$$\boxed{P(y) = \sum_{x} P(x,y)}$$

Conditional Probability :-

The conditional Probability of y given x is

$$\boxed{P(y/x) = \frac{P(x,y)}{P(x)}}$$

The conditional Probability of x given is

$$\boxed{P(x/y) = \frac{P(x,y)}{P(y)}}$$

1) If the Joined PMF of x, y is given by

$P(x,y) = K(2x+3y)$, $x = 0,1,2$; $y = 1,2,3$.

then Find

i) Find the value of K.

ii) Marginal distribution of x, y.

iii) conditional Probability of y on x.

iv) conditional Probability of x on y.

v) $P(x+y)$

vi) $P(x+y \leq 3)$

Soln:-

y\x	0	1	2
1	$3K = \frac{3}{72}$	$5K = \frac{5}{72}$	$7K = \frac{7}{72}$
2	$6K = \frac{6}{72}$	$8K = \frac{8}{72}$	$10K = \frac{10}{72}$
3	$9K = \frac{9}{72}$	$11K = \frac{11}{72}$	$13K = \frac{13}{72}$

i) WKT $\sum\sum P(x,y) = 1$

$3K + 5K + 7K + 6K + 8K + 10K + 9K + 11K + 13K = 1$

$72K = 1 \quad (\because) K = \frac{1}{72}$

(ii) The marginal distribution of x.

x	0	1	2
P(x)	$\frac{18}{72}$	$\frac{24}{72}$	$\frac{30}{72}$

Marginal distribution of y

y	P(y)
1	$15k = \frac{15}{72}$
2	$24k = \frac{24}{72}$
3	$33k = \frac{33}{72}$

iii) conditional Probability of y on x

$$P(y/x) = \frac{P(x,y)}{P(x)}$$

y\x	0	1	2
1	$3/18$ $\frac{3/72}{18/72}$	$5/24$	$7/30$
2	$6/18$	$8/24$	$10/30$
3	$9/18$	$11/24$	$13/30$

iv) conditional Probability of x on y

$$P(x/y) = \frac{P(x,y)}{P(y)}$$

$Y \backslash X$	0	1	2
1	$3/72$ / $15/72$ $3/15$	$5/15$	$7/15$
2	$6/24$	$8/24$	$10/24$
3	$9/33$	$11/33$	$13/33$

v) X : 0 1 2

Y : 1 2 3

(0, 1)
(0, 2) (1, 1)
(0, 3) (1, 2) (2, 1)

$X + Y$: 1 2 3 4 5

$x+y$:	1	2	3	4	5
$P(x+y)$:	$\frac{3}{72}$	$\frac{11}{72}$	$\frac{24}{72}$	$\frac{21}{72}$	$\frac{13}{72}$

vi) $P(X+Y > 3) = P(4, 5)$

$$= \frac{21}{72} + \frac{13}{72} = \frac{34}{72}$$

$$P(x+y > 3) = \frac{34}{72}$$

Two dimensional ~~condit~~ continuous Random Variable :-

Two Random variables x and y are continuous if the possible values of (x, y) ~~rise~~ lies in a region.

Joint PDF :-

Let x, y be two dimensional continuous random variable. Then, f(x, y) is called joint pdf. if

(i) $f(x,y) \geq 0$

ii) $\iint f(x,y)\,dx\,dy = 1$

Marginal PDF :-

The Marginal Distribution of x is

$$f(x) = \int f(x,y)\,dy$$

The Marginal Distribution of y is

$$f(y) = \int f(x,y)\,dx$$

Conditional Distribution :-

The conditional distribution of y on x is

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

The conditional distribution of x on y is

The Joint Pdf of x and y is given by

$$f(x,y) = \begin{cases} \lambda x y^2, & 0 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

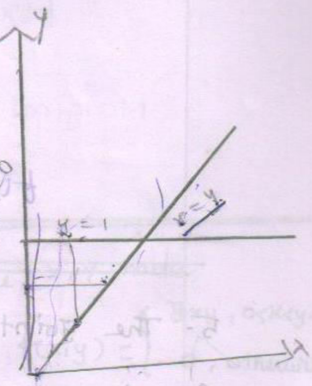
i) Determine λ.

ii) Find the marginal probability density function of x.

Soln:-

Given that;

$$f(x,y) = \begin{cases} \lambda x y^2, & 0 \le x \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$



WKT,

$$\iint f(x,y)\,dx\,dy = 1$$

$$\int_0^1 \int_0^y \lambda x y^2\,dx\,dy = 1$$

$$\int_0^1 \left[\frac{\lambda x^2 y^2}{2}\right]_0^y dy = 1$$

$$\int_0^1 \frac{\lambda y^4}{2}\,dy = 1$$

$$\frac{\lambda}{2}\left[\frac{y^5}{5}\right]_0^1 = 1$$

$$\frac{\lambda}{2}\left[\frac{1}{5}\right] = 1$$

$$\frac{\lambda}{10} = 1$$

$$\boxed{\lambda = 10}$$

The Marginal Pdf of X is

$$f(x) = \int f(x,y)\,dy$$

$$= \int_x^1 \lambda x y^2\,dy$$

$$= \lambda x \left[\frac{y^3}{3}\right]_x^1$$

$$= \lambda x \left[\frac{1}{3} - \frac{x^3}{3}\right]$$

$$= \frac{\lambda x(1-x^3)}{3}$$

$$f(x) = \frac{10x(1-x^3)}{3}$$

The Joint density Function of x and y is Given by

$$f(x,y) = \begin{cases} \frac{1}{2} y e^{-xy}, & 0<x<\infty,\ 0<y<2 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the conditional density Function of x given $y = 1$.

Soln:-

The conditional density fn of x given y $= f(x|y) = \frac{f(x,y)}{f(y)}$

$$f(y) = \int f(x,y)\,dx$$

$$= \int_0^\infty \frac{1}{2} y e^{-xy}\,dx$$

$$= \frac{1}{2} y \left[\frac{e^{-xy}}{-y}\right]_{x=0}^{\infty}$$

$$= \frac{1}{2} y \left[\frac{e^{-\infty}}{-y} - \frac{e^{0}}{-y}\right]$$

$$= \frac{1}{2} y \left[\frac{1}{y}\right] = \frac{1}{2}$$

$$f(x/y) = \frac{\frac{1}{2} y e^{-xy}}{+1/2}$$

$$= y e^{-xy}$$

$$f(x/y=1) = e^{-xy}$$

Two dimensional Random variable X, Y have Joint Probability density Function $f(x,y) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$

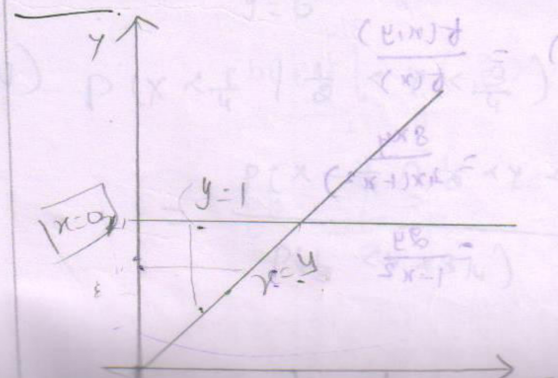
i) Find Marginal and conditional distributions

ii) x and y are independent

iii) Find $P(X < \frac{1}{2}, Y < \frac{1}{4})$

iv) Find $P(X < \frac{1}{4} \mid \frac{1}{2} < Y < \frac{3}{4})$

Soln:-



Marginal distribution of x

$$f(x) = \int f(x,y)\,dy$$

$$= \int_x^1 8xy\,dy$$

$$= 8x\left[\frac{y^2}{2}\right]_x^1$$

$$= 8x\left[\frac{1}{2} - \frac{x^2}{2}\right]$$

$$= \frac{8x\left[1-x^2\right]}{2}$$

$$f(x) = 4x(1-x^2)$$

Marginal distribution of y.

$$f(y) = \int f(x,y)\,dx$$

$$= \int_0^y 8xy\,dx$$

$$= 8y\left[\frac{x^2}{2}\right]_0^y$$

$$= 8y\left[\frac{y^2}{2}\right]$$

$$f(y) = 4y^3$$

The conditional distribution of y on x is

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{8xy}{4x(1-x^2)}$$

$$= \frac{2y}{1-x^2}$$

Correlation

Correlation is a measure of relationship between variables.

Positive correlation:-

Correlation coefficient:-

The correlation coefficient is given by

$$r = \frac{E[XY] - E[X] \cdot E[Y]}{\sqrt{\operatorname{var}(x) \cdot \operatorname{var}(y)}}$$

Note:-

i) $-1 \le r \le 1$

Procedure:-

i) Write the formula

ii) Find $E[XY]$

iii) Find $f(x)$ and $f(y)$

iv) Find $E[X]$ and $E[Y]$

v) Find $E[X^2]$ and $E[Y^2]$

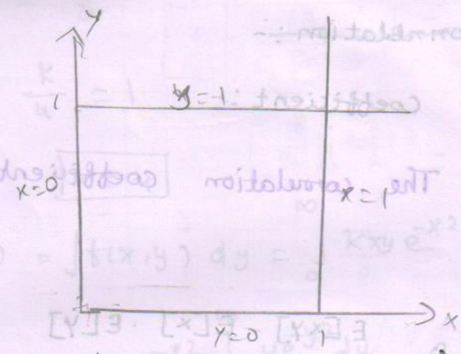
vi) Variance of x and variance of y.

Find the correlation coefficient for the two random variable X and Y having the joint density function

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), & 0 \le x \le 1;\ 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Soln:-

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$



$$r = \frac{E[xy] - E[x]\cdot E[y]}{\sqrt{var(x)\cdot var(y)}} \rightarrow \text{covariance}$$

$$E[xy] = \iint xy \, f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 xy \cdot \frac{3}{2}(x^2+y^2)\, dx\, dy$$

$$= \frac{3}{2}\int_0^1\int_0^1 (x^3y + xy^3)\, dx\, dy$$

$$= \frac{3}{2}\int_0^1 \left[\frac{x^4y}{4} + \frac{x^2y^3}{2}\right]_{x=0}^{1} dy$$

$$= \frac{3}{2}\int_0^1 \left(\frac{y}{4} + \frac{y^3}{2}\right) dy$$

$$= \frac{3}{2}\left[\frac{y^2}{8} + \frac{y^4}{8}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{8} + \frac{1}{8}\right]$$

$$= \frac{3}{2} \times \frac{2}{8}$$

$$\boxed{E[xy] = \frac{3}{8}}$$

$$f(x) = \int f(x,y)\, dy$$

$$= \int_0^1 \frac{3}{2}(x^2+y^2)\, dy$$

$$= \frac{3}{2}\left[x^2y + \frac{y^3}{3}\right]_{y=0}^{1}$$

$$= \frac{3}{2}\left[x^2 + \frac{1}{3}\right]$$

$$\boxed{f(x) = \frac{3x^2}{2} + \frac{1}{2}}$$

$$E[X] = \int x \cdot f(x)\, dx$$

$$= \int_0^1 \left(\frac{3x^3}{2} + \frac{x}{2}\right) dx$$

$$= \left(\frac{3x^4}{8} + \frac{x^2}{2}\right)_0^1$$

$$= \frac{3}{8} + \frac{1}{4}$$

$$\boxed{E[X] = \frac{5}{8}}$$

$$E[X^2] = \int x^2 f(x)\, dx$$

$$= \int_0^1 \left(\frac{3x^4}{2} + \frac{x^2}{2}\right) dx$$

$$= \left[\frac{3x^5}{10} + \frac{x^3}{6}\right]_0^1$$

$$= \frac{3}{10} + \frac{1}{6}$$

$$\boxed{E[X^2] = \frac{28}{60}}$$

Mean $= E[X] = \frac{5}{8}$

$$f(y) = \int f(x,y)\, dx$$

$$= \int_0^1 \frac{3}{2}(x^2+y^2)\, dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3} + y^2x\right]_{x=0}^{1}$$

$$= \frac{3}{2}\left[\frac{1}{3} + y^2\right]$$

$$\boxed{f(y) = \frac{1}{2} + \frac{3}{2}y^2}$$

$$E[Y] = \int y \cdot f(y)\, dy$$

$$= \int_0^1 \left(\frac{y}{2} + \frac{3}{2}y^3\right) dy$$

$$= \left(\frac{y^2}{4} + \frac{3}{8}y^4\right)_0^1$$

$$= \frac{1}{4} + \frac{3}{8}$$

$$\boxed{E[Y] = \frac{5}{8}}$$

$$E[Y^2] = \int y^2 f(y)\, dy$$

$$= \int_0^1 \left(\frac{y^2}{2} + \frac{3}{2}y^4\right) dy$$

$$= \left(\frac{y^3}{6} + \frac{3y^5}{10}\right)_0^1$$

$$= \frac{1}{6} + \frac{3}{10}$$

$$\boxed{E[Y^2] = \frac{28}{10}}$$

Mean $= E[Y] = 5$

$$Var(x) = E[X^2] - mean^2$$

$$= \frac{28}{60} - \frac{25}{64}$$

$$\boxed{Var(x) = \frac{73}{960}}$$

$$Mean = E[Y] = \frac{5}{8}$$

$$Var(y) = E[y^2] - Mean^2$$

$$= \frac{28}{60} - \frac{25}{64}$$

$$\boxed{Var(y) = \frac{73}{960}}$$

$$\therefore r = \frac{\frac{3}{8} - \frac{5}{8} \times \frac{5}{8}}{\sqrt{\frac{73}{960} \times \frac{73}{960}}}$$

$$= \frac{-1}{64} \times \frac{960}{73}$$

$$\boxed{r = \frac{-15}{73}}$$

Find the correlation coefficient x and y

x	1	3	5	7	8	10
y	8	12	15	17	18	20

Soln:-

$$r = \frac{E[xy] - E[x] \cdot E[y]}{\sqrt{var(x) \cdot var(y)}}$$

$$E[xy] = \frac{\Sigma xy}{n} = \frac{582}{6} = 97$$

$$E[x] = \frac{\Sigma x}{n} = \frac{34}{6} = 5.66$$

$$E[y] = \frac{\Sigma y}{n} = \frac{90}{6} = 15$$

$$var(x) = E[x^2] - mean^2 = \frac{\Sigma x^2}{n} - mean^2 = \frac{248}{6} - (5.66)^2$$

$$= 41.3 - (5.66)^2$$

$$= 9.2944.$$

$$var(y) = E[y^2] - mean^2 = \frac{\Sigma y^2}{n} - mean^2 = \frac{1446}{6} - (15)^2$$

$$= 241 - 225$$

$$= 16$$

$$r = \frac{E[xy] - E[x] \cdot E[y]}{\sqrt{var(x) \cdot var(y)}}$$

$$= \frac{97 - (5.66)(15)}{\sqrt{(9.2944)(16)}}$$

$$\boxed{r = 0.9878}$$

Regression

Regression is the functional relationship between the variables.

FORMULA:-

i) The regression Equation of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

where, $b_{yx} = \dfrac{r\sigma_y}{\sigma_x}$

ii) The regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

where, $b_{xy} = \dfrac{r\sigma_x}{\sigma_y}$

$\bar{x} \rightarrow$ Mean of x. (ie) $E[X]$

$\bar{y} \rightarrow$ mean of y (ie) $E[Y]$

$\sigma_y \rightarrow$ standard deviation of y $= \sqrt{var(x)}$

$\sigma_x \rightarrow$ standard deviation of x $= \sqrt{var(x)}$

Here,

$b_{yx}, b_{xy} \rightarrow$ Regression coefficient

NOTE :-

1. Two regression lines intersect at $(\bar{x}, \bar{y})$

2. $b_{yx} \cdot b_{xy} = r^2$

3. b_{xy}, b_{yx}, r have the same sign

4. $b_{xy} > 1 \Rightarrow b_{yx} < 1$

NOTE :-

If θ is the angle between the regression lines then

$$\tan\theta = \left(\frac{1-r^2}{r}\right)\cdot\left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$

If $r = 0$, then the regression lines are at right angles ($\perp r$)

If $r = \pm 1$, then the regression lines co-inside

Obtain the regression lines from the following data

X	1	2	3	4	5	6	7
Y	9	8	10	12	11	13	14

Soln :-

Regression line of Y on X :

$y - \bar{y} = b_{yx}(x - \bar{x})$ where $b_{yx} = \dfrac{r\sigma_y}{\sigma_x}$ $\longrightarrow$ (1)

Regression line of x on y

$$x-\bar{x} = b_{xy}(y-\bar{y}) \quad \text{where } b_{xy} = \frac{r\sigma_x}{\sigma_y} \rightarrow ②$$

$$r = \frac{E[XY] - E[X]\cdot E[Y]}{\sqrt{\mathrm{var}(x)\cdot \mathrm{var}(y)}}$$

$$E[XY] = \frac{\Sigma xy}{n} = \frac{334}{7} = 47.7142$$

$$E[X] = \frac{\Sigma x}{n} = \frac{28}{7} = 4.$$

$$E[Y] = \frac{\Sigma y}{n} = \frac{77}{7} = 11$$

$$\mathrm{var}(x) = E[x^2] - \mathrm{Mean}^2 = \frac{\Sigma x^2}{n} - \mathrm{mean}^2$$

$$= \frac{140}{7} - 4^2 = 4$$

$$\mathrm{var}(y) = E[y^2] - \mathrm{Mean}^2 = \frac{\Sigma y^2}{n} - \mathrm{mean}^2$$

$$= \frac{875}{7} - 11^2 = 4.$$

$$r = \frac{47.7142 - 4\times 11}{\sqrt{4\times 4}} = \frac{3.7142}{4} = 0.9285$$

$$\bar{x} = 4;\ \bar{y} = 11$$

$$\sigma_x = \sqrt{\mathrm{var}(x)} = 2;\quad \sigma_y = \sqrt{\mathrm{var}(y)} = 2$$

$$b_{yx} = \frac{r\sigma_y}{\sigma_x} = \frac{0.9285\times \cancel{2}}{\cancel{2}} = 0.9285$$

$$b_{xy} = \frac{r\sigma_x}{\sigma_y} = \frac{0.9285\times 2}{2} = 0.9285$$

$$(1) \Rightarrow y - 11 = 0.9285\,(x - 4)$$

$$\Rightarrow y - 4 = 0.9285x - 3.714 + 4$$

$$\boxed{y = 0.9285x + 7.286}$$

$$(2) \Rightarrow x - 4 = 0.9285\,(y - 11)$$

$$x - 4 = 0.9285y - 10.2135$$

$$\Rightarrow \boxed{x = 0.9285y - 6.2135}$$

The two lines of regression are $8x - 10y + 66 = 0$, $40x - 18y - 214 = 0$. The variance of x is 9.

i) Find mean values of x and y.

ii) Find the correlation coefficient of x and y.

iii) find the angle between the regression lines

Soln:-

Given,

$$8x - 10y = -66 \rightarrow (1)$$
$$40x - 18y = 214 \rightarrow (2)$$
$$var(x) = 9$$

$$(2) \Rightarrow 40x - 18y = 214$$

$$(5) \times (1) \Rightarrow 40x - 50y = -330$$

$$32y = 544.$$

$$\boxed{y = 17}$$

$① \Rightarrow 8x - 170 = -66$

$\Rightarrow 8x = 104$

$$\boxed{x = 13}$$

Mean of x is 13

Mean of y is 17

$8x - 10y = -66 \Rightarrow -10y = -8x - 66$

$$y = \frac{-8}{-10}x - \frac{66}{-10}$$

$$y = \frac{8}{10}x + \frac{66}{10}$$

$$\boxed{b_{yx} = \frac{8}{10}}$$

$40x - 18y = 214$

$40x = 18y + 214$

$$x = \frac{18}{40}y + \frac{214}{40}$$

$$\boxed{b_{xy} = \frac{18}{40}}$$

WKT, $r^2 = b_{xy} \cdot b_{yx}$

$$= \frac{18}{40} \times \frac{8}{10}$$

$$= \frac{9}{25}$$

$r^2 = \frac{9}{\ }$

$$\boxed{r = \frac{3}{5}}$$

$$\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

Given $var(x) = 9$

$$\sigma_x = \sqrt{var(x)} = 3$$

$$b_{yx} = \frac{8}{10}$$

$$r\frac{\sigma_y}{\sigma_x} = \frac{8}{10}$$

$$\frac{3}{5} \times \frac{\sigma_y}{3} = \frac{8}{10}$$

$$\sigma_y = \frac{8}{10} \times 5 = 4$$

$$\boxed{\sigma_y = 4}$$

$$\tan\theta = \frac{\left(1-\frac{9}{25}\right)}{3/5} \times \left(\frac{3 \times 4}{9+16}\right)$$

$$= \frac{16}{25} \times \frac{5}{3} \times \frac{12}{25}$$

$$\tan\theta = \frac{64}{125}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{64}{125}\right)}$$

Transformation of random variable.

Let X be a random variable with pdf $f(x)$. take $y = g(x)$

1. Find x interms of y.
2. Find $\left|\frac{dx}{dy}\right|$
3. $f(y) = \left|\frac{dx}{dy}\right| \cdot f(x)$
4. By using the range of x, find the range of y.

If the pdf of x is $f(x) = 2x,\ 0 < x < 1$ Find the pdf of $y = 3x+1$.

soln:-

Given,

$$f(x) = 2x, \quad 0 < x < 1$$

$$y = 3x+1$$

$$3x = y-1$$

$$x = \frac{y}{3} - \frac{1}{3}$$

$$\frac{dx}{dy} = \frac{1}{3}$$

$$\left|\frac{dx}{dy}\right| = \frac{1}{3}$$

$$f(y) = \left|\frac{dx}{dy}\right| f(x)$$

$$= \frac{1}{3} \cdot 2x = \frac{2}{3}x.$$

$$= \frac{2}{3}\left(\frac{y}{3} - \frac{1}{3}\right)$$

$$f(y) = \frac{2y}{9} - \frac{2}{9}$$

$0 < x < 1$

$\Rightarrow 0 < 3x < 3$

$\Rightarrow 1 < 3x+1 < 4$

$\Rightarrow 1 < y < 4$

Given the random variable x with density function $f(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$. Find the Pdf of $Y = 8x^3$

Soln:-

Given, $f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$

$Y = 8x^3$

$x^3 = \frac{y}{8}$

$$\boxed{x = \frac{y^{1/3}}{2}}$$

$$\frac{dx}{dy} = \frac{1}{3} \cdot \frac{y^{-2/3}}{2}$$

$$\left|\frac{dx}{dy}\right| = \frac{1}{6} \cdot y^{-2/3}$$

$$f(y) = \left|\frac{dx}{dy}\right| f(x)$$

$$= \frac{1}{6} \cdot y^{-2/3} \cdot 2x$$

$$= \frac{1}{6} \cdot y^{-2/3} \cdot 2\frac{y^{1/3}}{2}$$

$$\boxed{f(y) = \frac{1}{6} \; y^{-1/3}}$$

$$0 < x < 1$$

$$\Rightarrow 0 < x^3 < 1$$

$$\Rightarrow 0 < 8x^3 < 8$$

$$\Rightarrow 0 < y < 8$$

If $f(x) = e^{-x}$, $x > 0$, then find the pdf of $y = 2x - 1$.

<u>soln</u>:-

Given, $f(x) = e^{-x}$, $x > 0$

$$y = 2x - 1$$

$$2x = y + 1$$

$$x = \frac{y}{2} + \frac{1}{2}$$

$$\frac{dx}{dy} = \frac{1}{2}$$

$$\left|\frac{dx}{dy}\right| = \frac{1}{2}$$

$$f(y) = \left|\frac{dx}{dy}\right| f(x)$$

$$= \frac{1}{2} \cdot e^{-x}$$

$$= \frac{e^{-x}}{2}$$

$$f(y) = \frac{1}{2} \cdot e^{-(y/2 + 1/2)}$$

$$= \frac{1}{2} \cdot e^{-y/2} \cdot e^{-1/2}$$

$x > 0$

$2x > 0$

$2x - 1 > -1$

$y > -1$

Transformation of two random variables

i) Let x, y be two random variables whose joint pdf is $f(x,y)$.

ii) take $U = g(x,y)$ & $V = h(x,y)$

iii) Find x and y interms U and V

iv) Find Modulus of $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

v) $g(u,v) = |J| \cdot f(x,y)$

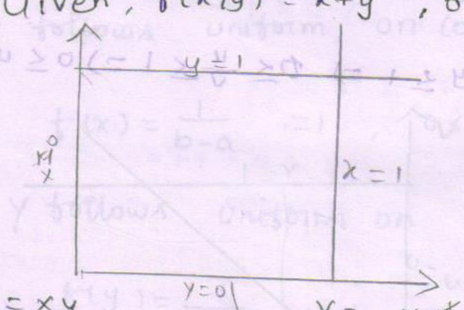
vi) By using the range of x and y,

Find the range of U and V.

If the joint pdf of x and y is given by $f(x,y) = x+y$, $0 \le x, y \le 1$ then find the pdf of xy.

Soln:-

Given; $f(x,y) = x+y$, $0 \le x$, $y \le 1$



$U = xy$ $V = x$

$U = vy$ $x = v$

$y = \frac{u}{v} = uv^{-1}$

$$\frac{\partial y}{\partial u} = v^{-1} \qquad \frac{\partial x}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = -uv^{-2} \qquad \frac{\partial x}{\partial v} = 1$$

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = 0 - \frac{1}{v} = -\frac{1}{v}$$

$$g(u,v) = |J| \cdot f(x,y)$$

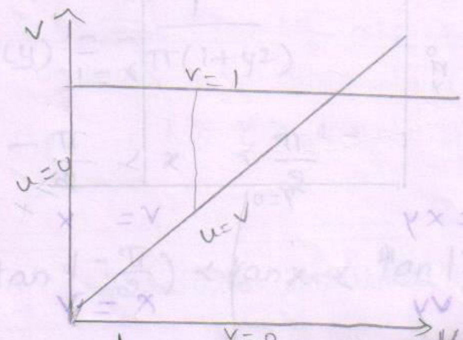
$$= \frac{1}{v} \cdot (x+y)$$

$$= \frac{1}{v}\left(v + \frac{u}{v}\right)$$

$$g(u,v) = 1 + \frac{u}{v^2} = 1 + uv^{-2}$$

$0 \le x \le 1 \Rightarrow 0 \le v \le 1$

$0 \le y \le 1 \Rightarrow 0 \le \frac{u}{v} \le 1 \Rightarrow 0 \le u \le v$



u	0	1
v	0	1

$$g(u) = \int g(u,v)\, dv$$

$$g(u) = \int_u^1 (1+uv^{-2})\,dv$$

$$= \left[v + u \cdot \frac{v^{-1}}{-1} \right]_{v=u}^{1}$$

$$= \left[1 + \frac{u}{-1} - \left(u + \frac{1}{-1} \right) \right]$$

$$= \left[1 - u - u + 1 \right]$$

$$= 2 - 2u$$

$$\boxed{g(u) = 2(1-u)}$$

Let x and y be independent random variables both uniformly distributed on (0,1). Calculate the Probability density function of x+y,

Soln:-

x follows uniform on (0,1)

$$f(x) = \frac{1}{b-a} = 1, \quad 0 < x < 1$$

y follows uniform on (0,1)

$$f(y) = \frac{1}{b-a} = 1, \quad 0 < y < 1$$

x and y are independent.

$$\Rightarrow f(x,y) = f(x) \cdot f(y)$$

$$f(x,y) = 1, \quad 0 < x \le 1, \quad 0 \le y < 1$$

$u = x+y \qquad v = x$

$u = v+y \qquad x = v$

$y = u-v$

$\frac{\partial y}{\partial u} = 1 \qquad \frac{\partial x}{\partial u} = 0$

$\frac{\partial y}{\partial v} = -1 \qquad \frac{\partial x}{\partial v} = 1$

$$J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0-1 = -1$$

$g(u,v) = |J| \cdot f(x,y)$

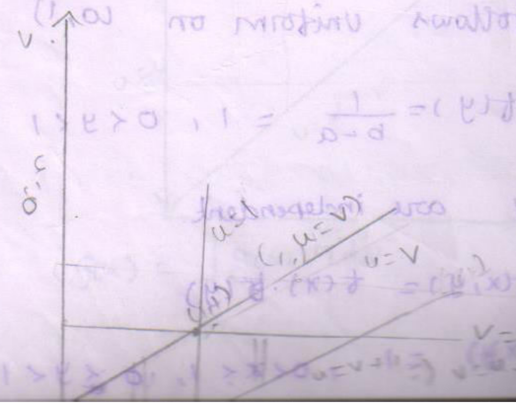
$= (1)\cdot(1)$

$= 1$

$0 < x < 1 \Rightarrow 0 < v < 1$

$0 < y < 1 \Rightarrow 0 < u-v < 1$

$\Rightarrow v < u < 1+v$



$u < 1$

$$g(u) = \int g(u,v)\, dv$$

$$= \int_0^u dv$$

$$= [v]_0^u$$

$$= u$$

$u > 1$

$$g(u) = \int g(u,v)\, dv$$

$$= \int_{u-1}^{1} dv$$

$$= [v]_{u-1}^{1}$$

$$= 1 - (u-1)$$

$$= 1 - u + 1$$

$$= 2 - u$$

$$g(u) = \begin{cases} u, & u < 1 \\ 2-u, & u > 1 \end{cases}$$